Kevin McIlhany, Reza Malek-Madani

Chesapeake Bay Analysis using Time and Spatial Generalized Eigenfunctions

Abstract A Normal Mode Analysis of the Chesapeake Bay has reached the phase of performing time-series analysis. Prior attempts using generalized eigenfunctions obtained from COMSOL have led to extraction of power spectra using a partial domain extraction and a wavelet-like correction to the result. A new approach has been developed which effectively solves the spatial equivalent of the Initial Value Problem. By taking data sets at a limited number of points (about 10) over extended periods of time, the Chesapeake Bays flow vector field can be extracted from the power spectra. This method combines the spatial modes derived from COMSOL with the generalized basis set for time. By overlaying windows of the spectra obtained, a progression of the spectral components is obtained for the lowest 100 eigenmodes. This progression can exhibit smooth behavior which allows for prediction of the modal behavior to take place, effectively predicting the flow across the entire Bay while only sampling the Bay at 10 locations.

Keywords Normal Mode Analysis · Harmonic Analysis · EOF · NMA · Chesapeake Bay · COMSOL MultiPhysics

1 Introduction

The study of vector flow fields for fluids has a rich history. A study of the Chesapeake Bays flow has been underway for several years by many different parties. Traditional approaches involve modeling the Bay in terms of fluid models coupled with geophysical effects such as the Coriolis force. The approach taken by the authors is not to model the Chesapeake Bays flow, merely to characterize its signal properly so that a model may emerge from the data. By changing the emphasis from model validation to data-driven signal characterization, a more focused effort is being made to understand the specific problems the data presents, irrespective of what physical models may or may not be relevant. The method of choice has been to calculate Normal Modes using COMSOL. For a historical review of Normal Mode Analysis (NMA), please refer to the work based on [Eremeev et al. 1992][1] [2] and [Lipphardt et al. 2000][3].

As a brief reminder, the two studies leading up to the Chesapeake Bay analysis were presented at COMSOL '05 and the World Congress on Computing CSC'06 conferences showing the Dirichlet [6], [9] and Neumann [7], [9] solutions computed via COMSOL as well as a finite difference in-house code. The numerical aspects of this paper were motivated by a method for completing surface current velocity fields called Normal Mode Analysis (NMA) [Eremeev et al. 1992][1] [2],[Lipphardt et al. 2000][3].

In each of the former cases studied using NMA the end goal was to be able to use Normal Modes to"fill in" vector fields where data is not present and to extract power spectra in order to search for time-dependent features. When strong time-dependent features are present, short-term prediction may be employed. Eremeev et al. studied the Black Sea, collecting data from autonomous drifting buoys (ADB). Lipphardt et al. used this same method to fill in gaps of velocity fields for Monterey Bay using HF radar for the data set. The Black Sea is a closed water body roughly in the shape of a kidney. Monterey Bay is open to the Pacific Ocean and has a shore roughly hemispherical. Lipphardt's group extended the NMA by adding a mode to account for the flow between Monterey Bay and the Pacific. The Chesapeake Bay has 11,684 miles of shoreline but is only 189 miles long by 30 miles wide, giving it a jagged shore, almost fractal, compounded by a large opening to the Atlantic Ocean at its southern end. The length of the Chesapeake Bay allows for both fresh water as well as salt water to exist within its boundary. As a result, the Chesapeake Bay represents a difficult system to model. Figures 1, 2

K. McIlhany (presenter)
Physics Dept.
Tel.: 410-293-6667
Fax: 410-293-3729
E-mail: mcilhany@usna.edu

R. Malek-Madani
Mathematics Dept.
Tel.: 410-293-2504
Fax: 410-293-2507
E-mail: rmm@usna.edu
United States Naval Academy
572 Holloway Rd.
Annapolis, MD 21402

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14. ABSTRACT
A Normal Mode Analysis of the Chesapeake Bay has reached the phase of performing time-series analysis. Prior attempts using generalized eigenfunctions obtained from COMSOL have led to extraction of power spectra using a partial domain extraction and a waveletlike correction to the result. A new approach has been developed which effectively solves the spatial equivalent of the Initial Value Problem. By taking data sets at a limited number of points (about 10) over extended periods of time the Chesapeake Bays flow vector field can be extracted from the power spectra. This method combines the spatial modes derived from COMSOL with the generalized basis set for time. By overlaying windows of the spectra obtained, a progression of the spectral components is obtained for the lowest 100 eigenmodes. This progression can exhibit smooth behavior which allows for prediction of the modal behavior to take place, effectively predicting the flow across the entire Bay while only sampling the Bay at 10 locations.

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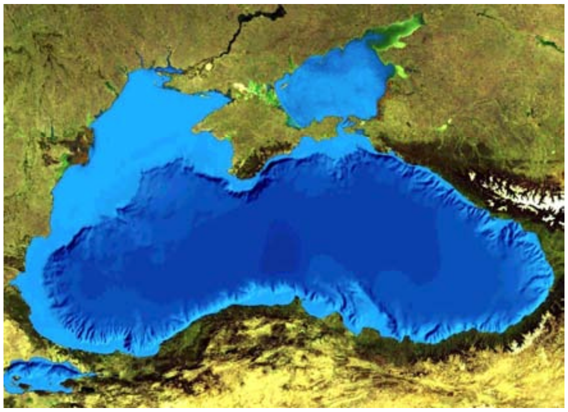


Fig. 1 The Black Sea compliments of NASA World Wind.

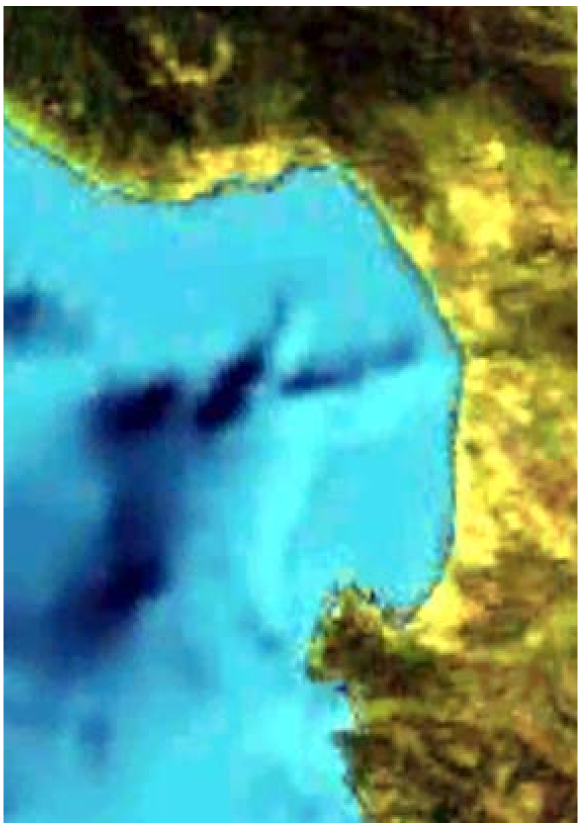


Fig. 2 Monterey Bay from NASA World Wind. The open boundary with the Pacific clearly visible.

and 5 illustrate the varying levels of complexity that have been solved using NMA.

The basic unit of calculation used throughout this paper is the normal mode. Like the modes of a guitar string or an organ pipe, systems obeying the Helmholtz equation and Dirichlet or Neumann boundary conditions will resonate in states referred as "normal modes". For the Chesapeake Bay, the modes calculated are energy potentials whose gradients and curls of gradients correspond to the vector current fields found in fluid mechanics ($\overrightarrow{u}$).

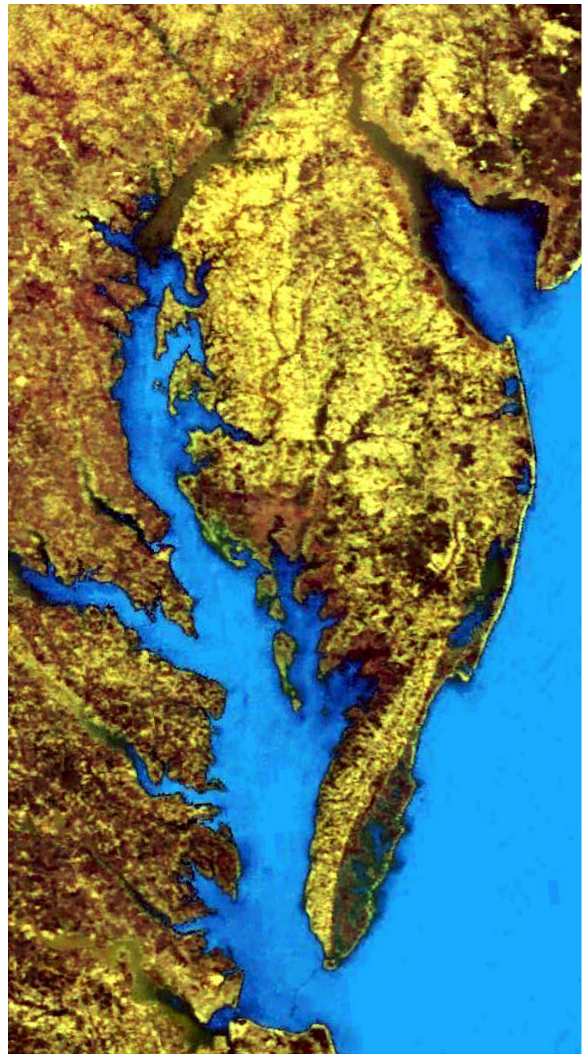


Fig. 3 Chesapeake Bay from NASA World Wind. The Atlantic ocean open at the southern end, allowing in salt water. The north end dominated by fresh water.

Briefly, the formulation leading to the calculation of fluid flow stems from the realization that the vector fields can be derived from two scalar fields, which are the solutions to the Helmhotz equation under Dirichlet and Neumann boundary conditions [4].

$$\overrightarrow{u} = \nabla \times [(\hat{n}\Psi) + \nabla \times (\hat{n}\Phi)]. \tag{1}$$

Here Ψ is the stream potential where,

$$\overrightarrow{u}_D = (u,v)_D = \left(\frac{-\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial x} \right), \tag{2}$$

and Φ is the velocity potential where,

$$\overrightarrow{u}_N = (u,v)_N = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right), \tag{3}$$

with (u,v) representing the surface current velocities in the x and y directions respectively. The total velocity field is composed:

$$\vec{u}(x,t) = \sum_{n=0}^{\infty} \left[a(t)_n \vec{u}_{D,n} + b(t)_n \vec{u}_{N,n}\right] + \vec{u}_{Source}. \quad (4)$$

The non-conserving flow of mass across the boundary, $\vec{u}_{Source}$, is model or experimental data of the fluid velocity at the boundary. The source term requires input from an external reference, in this case, the work of Tom Gross at NOAA was employed where source currents were provided from the QUODDY finite element model of the Bay. Although QUODDY is a model of the Chesapeake Bay's currents, it draws data from multiple sets collected near the mouth of the Bay (the Atlantic ocean interface) as well as selected points from the interior of the Bay's geometry. For a recent review of efforts by NOAA and Tom Gross, see reference [5].

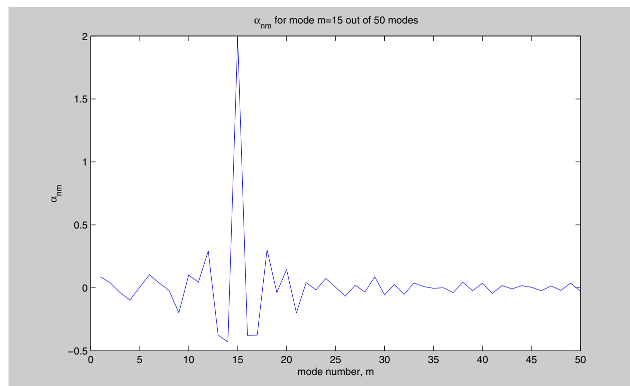
2 Time Series Analysis

The Galerkin method is used to calculate the amplitude for a given normal mode. A key point to this analysis is that the technique requires knowledge of the data over the full domain. The source term will be subtracted from the outset, eliminating it from this analysis. Only focusing on the x-component of the vector flow field, $u(x,t)$, and assuming a data set exists which spans the entire x-domain, $u(x,t)_{data}$,

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} \left[a(t)_n u(x)_{D,n} + b(t)_n u(x)_{N,n}\right] \\
a(t)_m &= \oint u(x,t)_{data} u(x)_{D,m} d\Omega \\
a(t)_m &= \oint \sum_{n=0}^{\infty} \left[a(t)_n u(x)_{D,n} + b(t)_n u(x)_{N,n}\right] u(x)_{D,m} d\Omega \\
a(t)_m &= \sum_{n=0}^{\infty} \left[a(t)_n \oint u_{D,n} u_{D,m} d\Omega + b(t)_n \oint u_{N,n} u_{D,m} d\Omega\right] \\
a(t)_m &= \sum_{n=0}^{\infty} \left[a(t)_n \delta_{nm} + b(t)_n \o\right] \\
a(t)_m &= \delta_{nm} a(t)_n . \\
a(t)_n &= \oint u(x,t)_{data} u(x)_{D,n} d\Omega
\end{aligned}$$

similarly, for the Neumann coefficients, b_n, coefficients are calculated by integrating with the Neumann solutions, $u(x)_{N,n}$ over the domain.

Central to the theme of this derivation is the application of orthonormality of the basis sets. The common reason for not applying this technique is that full knowledge of the vector field is required over the entire domain in order to obtain the coefficients $a(t)_n$ and $b(t)_n$. For a waterway like the Chesapeake Bay, it is impractical to assume that the entire vector flow field can ever be obtained at any one time. Satellite data may be able to provide information related to vector flow, however, in order to extract useful time series would require long-term surveillance of the area in question. Satellite data is simply too expensive. At the COMSOL 2006 conference, a partial domain Galerkin method was investigated and found to yield wavelet-like responses in the spectral domain if the data did not cover 100% of the domain. This effect is partially correctable, but unwanted. In the worst case scenario, the results are unusable [8]. Figure 2 shows the response of the power spectrum for the 15th mode when only 50% of the domain has data. For a delta function, the response is similar to the function $sinc(k\,x) = \frac{sin(k\,x)}{k\,x}$, prevelant in wavelet theory.



3 Normal Mode Analysis

Taking a step back, in solving a 1D wave-like problem with both spatial and temporal boundary conditions may reveal a different approach. Consider a guitar string of length, L. Held at both ends, clearly a Dirichlet problem. Using the wave equation:

$$c^2\nabla^2 F(x,t) = \frac{\partial^2}{\partial t^2} F(x,t), \qquad F(x,t) = f(x)\,g(t). \quad (5)$$

Rearrangement of the equation leads to a separation of variables:

$$\frac{\partial^2}{\partial x^2} f(x)g(t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x)g(t). \quad (6)$$

$$\frac{f_{xx}}{f(x)} = \frac{g_{tt}}{c^2 g(t)} = -\lambda. \quad (7)$$

leading to the two separate Helmholtz equations:

$$f(x)_{xx} = -\lambda f(x). \quad (8)$$

$$g(t)_{tt} = -c^2\lambda g(t). \quad (9)$$

After applying boundary conditions

$$\nabla^2 f(x) = -\lambda f(x)|_{\partial D}. \quad (10)$$

$$f(x)_{D,n} = \sqrt{\tfrac{2}{L}}\, sin(k_n\, x), \qquad k_n = \tfrac{2\,\pi}{\lambda_n} = \tfrac{n\pi}{L} \tag{11}$$

where λ represents the eigenvalue and λ_n is the wavelength. Similarly for time:

$$\frac{\partial^2}{\partial t^2} g(t) = -c^2\, \lambda g(t)|_{\partial D}, \qquad c = \frac{L}{T}. \tag{12}$$

$$g(t)_{D,n} = \sqrt{\tfrac{2}{T}}\, sin(\omega_n\, t), \qquad \omega_n = c\, k_n \tag{13}$$

This formulation leads to the well-known "Initial Value Problem (IVP)" from standard differential equation books. Namely, if a function is known completely across the x-axis at $t = 0 \equiv t_0$ and its time-derivative is also known for all x at t_0, the complete Fourier spectra can be determined for all time. To illustrate this solution, consider the case when there are mixed boundaries in both space and time. Under those circumstances, the general solution is formed by allowing both Dirichlet and Neumann solutions to exist. The basis functions, $f(x)$ and $g(t)$ are known from the underlying differential equations. In this case, the general solution takes the form (neglecting any source terms):

$$F(x,t) = \sum_{n=0}^{\infty} \left[A_n g(t)_{D,n} + B_n g(t)_{N,n}\right]\left[C_n f(x)_{D,n} + D_n f(x)_{N,n}\right]$$

$$\begin{aligned}
F(x,t_0) = \sum_{n=0}^{\infty} & \left[AC_n g(t_0)_{D,n} f(x)_{D,n} + BC_n g(t_0)_{N,n} f(x)_{D,n}\right] + \\
& + \left[AD_n g(t_0)_{D,n} f(x)_{N,n} + BD_n g(t_0)_{N,n} f(x)_{N,n}\right]
\end{aligned}$$

$$\alpha_m = \oint F(x,t_0)_{data} f(x)_{D,m}\, d\Omega$$

$$\begin{aligned}
\alpha_m = \sum_{n=0}^{\infty} & \oint \left[AC_n g(t_0)_{D,n} f(x)_{D,n}\right] f(x)_{D,m}\, d\Omega \; + \\
& + \oint \left[BC_n g(t_0)_{N,n} f(x)_{D,n}\right] f(x)_{D,m}\, d\Omega \; + \\
& + \oint \left[AD_n g(t_0)_{D,n} f(x)_{N,n}\right] f(x)_{D,m}\, d\Omega \; + \\
& + \oint \left[BD_n g(t_0)_{N,n} f(x)_{N,n}\right] f(x)_{D,m}\, d\Omega
\end{aligned}$$

$$\begin{aligned}
\alpha_m = \sum_{n=0}^{\infty} & \left[AC_n g(t_0)_{D,n} + BC_n g(t_0)_{N,n}\right] \oint f_{D,n} f_{D,m}\, d\Omega \\
& + \left[AD_n g(t_0)_{D,n} + BD_n g(t_0)_{N,n}\right] \oint f_{N,n} f_{D,m}\, d\Omega
\end{aligned}$$

$$\begin{aligned}
\alpha_m = \sum_{n=0}^{\infty} & \left[AC_n g(t_0)_{D,n} + BC_n g(t_0)_{N,n}\right] \delta_{nm} \; + \\
& + \left[AD_n g(t_0)_{D,n} + BD_n g(t_0)_{N,n}\right] \emptyset
\end{aligned}$$

$$\alpha_n = \left[AC_n g(t_0)_{D,n} + BC_n g(t_0)_{N,n}\right]$$

$$\alpha_n = \oint F(x,t_0) f(x)_{D,n}\, d\Omega$$

similarly for the Neumann term:

$$\beta_m = \oint F(x,t_0)_{data} f(x)_{N,m}\, d\Omega$$

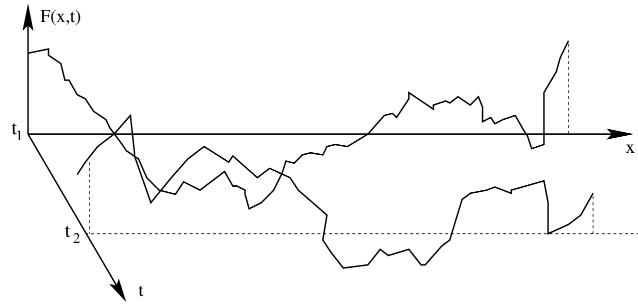


Fig. 4 Dual Time Problem: at two times, t_1 and t_2, the function $F(x,t)$ is completely known along the x-axis.

$$\begin{aligned}
\beta_m = \sum_{n=0}^{\infty} & \oint \left[AC_n g(t_0)_{D,n} f(x)_{D,n}\right] f(x)_{N,m}\, d\Omega \; + \\
& + \oint \left[BC_n g(t_0)_{N,n} f(x)_{D,n}\right] f(x)_{N,m}\, d\Omega \; + \\
& + \oint \left[AD_n g(t_0)_{D,n} f(x)_{N,n}\right] f(x)_{N,m}\, d\Omega \; + \\
& + \oint \left[BD_n g(t_0)_{N,n} f(x)_{N,n}\right] f(x)_{N,m}\, d\Omega
\end{aligned}$$

$$\begin{aligned}
\beta_m = \sum_{n=0}^{\infty} & \left[AC_n g(t_0)_{D,n} + BC_n g(t_0)_{N,n}\right] \oint f_{D,n} f_{N,m}\, d\Omega \\
& + \left[AD_n g(t_0)_{D,n} + BD_n g(t_0)_{N,n}\right] \oint f_{N,n} f_{N,m}\, d\Omega
\end{aligned}$$

$$\begin{aligned}
\beta_m = \sum_{n=0}^{\infty} & \left[AC_n g(t_0)_{D,n} + BC_n g(t_0)_{N,n}\right] \emptyset \; + \\
& + \left[AD_n g(t_0)_{D,n} + BD_n g(t_0)_{N,n}\right] \delta_{nm}
\end{aligned}$$

$$\beta_n = \left[AD_n g(t_0)_{D,n} + BD_n g(t_0)_{N,n}\right]$$

$$\beta_n = \oint F(x,t_0) f(x)_{N,n}\, d\Omega$$

The problem with using mixed boundaries becomes apparent quickly. There are four unknowns, AC_n, BC_n, AD_n, and BD_n, yet only two equations of constraint coming from the initial conditions, namely from $F(x,t_0)$, we get α_n and β_n. Two more equations are needed in order to fully determine the amplitudes of the basis functions. The Initial Value Problem uses knowledge of two pieces of information, $F(x,t_0)$ and $\frac{\partial}{\partial t} F(x,t|_{t_0})$. To set up the problem attempted on the Chesapeake Bay, the IVP is altered to a new form, namely, the "Dual Time Problem", (DTP), where a second time is introduced instead of using $\frac{\partial}{\partial t} F(x,t|_{t_0})$.

4 Dual Time Problem

Given complete knowledge of a function in x at two times, $F(x,t_1)$ and $F(x,t_2)$, four quantities are defined, α_n, β_n, γ_n, and Δ_n. From these four, the spectral components are fully determined.

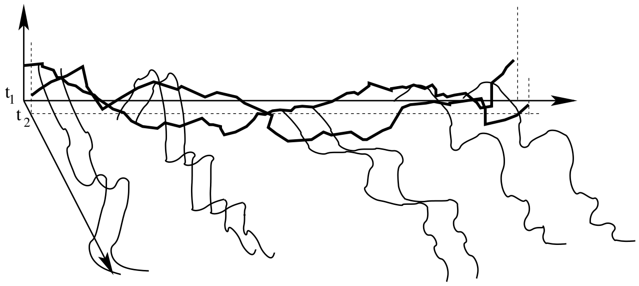


Fig. 5 Once the coefficients have been calculated, the solution may be moved forward in time, allowing for prediction.

$$\alpha_n = \oint F(x,t_1)_{data} f(x)_{D,n} d\Omega$$
$$\beta_n = \oint F(x,t_1)_{data} f(x)_{N,n} d\Omega$$
$$\gamma_n = \oint F(x,t_2)_{data} f(x)_{D,n} d\Omega$$
$$\Delta_n = \oint F(x,t_2)_{data} f(x)_{N,n} d\Omega$$

$$\alpha_n = \left[AC_n g(t_1)_{D,n} + BC_n g(t_1)_{N,n}\right]$$
$$\beta_n = \left[AD_n g(t_1)_{D,n} + BD_n g(t_1)_{N,n}\right]$$
$$\gamma_n = \left[AC_n g(t_2)_{D,n} + BC_n g(t_2)_{N,n}\right]$$
$$\Delta_n = \left[AD_n g(t_2)_{D,n} + BD_n g(t_2)_{N,n}\right]$$

$$\begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \Delta_n \end{pmatrix} = \begin{pmatrix} AC_n & BC_n \\ AD_n & BD_n \end{pmatrix} \begin{pmatrix} g(t_1)_{D,n} & g(t_2)_{D,n} \\ g(t_1)_{N,n} & g(t_2)_{N,n} \end{pmatrix}$$

$$\begin{pmatrix} AC_n & BC_n \\ AD_n & BD_n \end{pmatrix} = \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \Delta_n \end{pmatrix} \begin{pmatrix} g(t_1)_{D,n} & g(t_2)_{D,n} \\ g(t_1)_{N,n} & g(t_2)_{N,n} \end{pmatrix}^{-1}$$

Several important things to note about this result. This is really just the IVP masked over two times. The stablity of the solution depends on having good data covering the whole x-domain. Because the last matrix in the derivation must be inverted, it cannot be singular or close to singular. As a result, it is important to pick two times t_1 and t_2 that are not on any nodes for the temporal basis functions. Provided these conditions are met, the coefficients obtained should allow the function $F(x,t)$ to be projected ahead (or behind) in time. Finally, because the basis set was chosen in time to be *sines* and *cosines*, there is temporal repetition on the time scale of $T = L / c$. Outside of this time window, the temporal components repeat, preventing prediction further.

5 Dual Position Problem (conjugate to time problem)

The DTP is not new, nor is it controversial; it is simply an extension of the IVP. The goal of this paper is to present

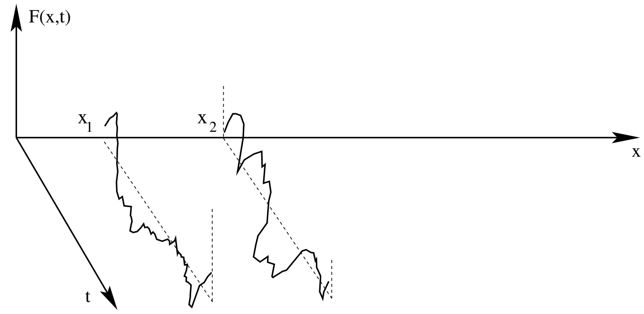


Fig. 6 Dual Position Problem: at two locations, x_1 and x_2, the function $F(x,t)$ is completely known along the t-axis.

the Dual Position Problem (DPP). Although a simple transformation from the time axis to the spatial axis, the results are contentious. Simply put, by collecting data at two spatial locations over a long enough time window, $\Delta t >= \Delta x_{max} / c$, the spectral components are determined and applied to the spatial domain, allowing spatial prediction.

$$F(x_1,t) = \sum_{n=0}^{\infty} \left[AC_n g(t)_{D,n} f(x_1)_{D,n} + BC_n g(t)_{N,n} f(x_1)_{D,n}\right] + \\ + \left[AD_n g(t)_{D,n} f(x_1)_{N,n} + BD_n g(t)_{N,n} f(x_1)_{N,n}\right]$$

$$\alpha_m = \oint F(x_1,t)_{data} g(t)_{D,m} d\,t$$

$$\alpha_m = \sum_{n=0}^{\infty} \oint \left[AC_n f(x_1)_{D,n} g(t)_{D,n}\right] g(t)_{D,m} d\,t \; + \\ + \oint \left[AD_n f(x_1)_{N,n} g(t)_{D,n}\right] g(t)_{D,m} d\,t \; + \\ + \oint \left[BC_n f(x_1)_{D,n} g(t)_{N,n}\right] g(t)_{D,m} d\,t \; + \\ + \oint \left[BD_n f(x_1)_{N,n} g(t)_{N,n}\right] g(t)_{D,m} d\,t$$

$$\alpha_m = \sum_{n=0}^{\infty} \left[AC_n f(x_1)_{D,n} + AD_n f(x_1)_{N,n}\right] \oint g_{D,n} g_{D,m} d\,t \\ + \left[BC_n f(x_1)_{D,n} + BD_n f(x_1)_{N,n}\right] \oint g_{N,n} g_{D,m} d\,t$$

$$\alpha_m = \sum_{n=0}^{\infty} \left[AC_n f(x_1)_{D,n} + AD_n f(x_1)_{N,n}\right] \delta_{nm} \; + \\ + \left[BC_n f(x_1)_{D,n} + BD_n f(x_1)_{N,n}\right] \emptyset$$

$$\alpha_n = \left[AC_n f(x_1)_{D,n} + AD_n f(x_1)_{N,n}\right]$$

$$\alpha_n = \int F(x_1,t)_{data} g(t)_{D,n} dt$$
$$\beta_n = \int F(x_1,t)_{data} g(t)_{N,n} dt$$
$$\gamma_n = \int F(x_2,t)_{data} g(t)_{D,n} dt$$
$$\Delta_n = \int F(x_2,t)_{data} g(t)_{N,n} dt$$

Fig. 7 Once the coefficients have been calculated, the solution may be used to solve across the entire spatial domain.

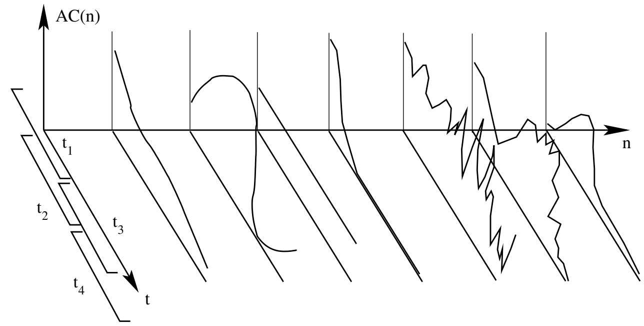


Fig. 8 By calculating overlapping windows in time, the spectra calculated are shown to exhibit time-dependence.

$$\begin{aligned}
\alpha_n &= AC_n f(x_1)_{D,n} + AD_n f(x_1)_{N,n} \\
\beta_n &= BC_n f(x_1)_{D,n} + BD_n f(x_1)_{N,n} \\
\gamma_n &= AC_n f(x_2)_{D,n} + AD_n f(x_2)_{N,n} \\
\Delta_n &= BC_n f(x_2)_{D,n} + BD_n f(x_2)_{N,n}
\end{aligned}$$

$$\begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \Delta_n \end{pmatrix} = \begin{pmatrix} AC_n & AD_n \\ BC_n & BD_n \end{pmatrix} \begin{pmatrix} f(x_1)_{D,n} & f(x_2)_{D,n} \\ f(x_1)_{N,n} & f(x_2)_{N,n} \end{pmatrix}$$

$$\begin{pmatrix} AC_n & AD_n \\ BC_n & BD_n \end{pmatrix} = \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \Delta_n \end{pmatrix} \begin{pmatrix} f(x_1)_{D,n} & f(x_2)_{D,n} \\ f(x_1)_{N,n} & f(x_2)_{N,n} \end{pmatrix}^{-1}$$

Similar to the DTP, several important things to note about this result. This is really just the DTP in space, not time. The stablity of the solution depends on having good data covering the whole t-domain. Because the last matrix in the derivation must be inverted, it cannot be singular or close to singular. As a result, it is important to pick two locations x_1 and x_2 that are not on any nodes for the spatial basis functions. Provided these conditions are met, the coefficients obtained should allow the function $F(x,t)$ to be projected in space. Finally, because the spatial basis set was computed in COMSOL, confidence in the solution is paramount. The whole derivation is hinged on orthonormality, so when COMSOL computes a "tight" set of functions, this method should work. For most complex boundaries, it is not practical to extend the solution beyond the stated spatial boundaries, however, by choosing a time window long enough to span the largest extant of the spatial domain, this issue should not be a problem.

6 Results and Conclusions

The coefficients computed in spectral domain are assumed to be constant in both space and time for the windows under consideration. Nature, however, is not static. By continuing to take data at spatial locations over time, overlapping time windows may be employed to re-calculate the coefficients. In the new time window, the coefficients will still be constants, however, they will represent the average spectral coefficients for that specific timing window. By successively overlapping time windows and computing spectral components, a time series for the spectra is obtained, as illustrated in figure 8. From this process, time prediction may be achieved by fitting the spectral time variation to traditional functions, polynomials, sinusoid, exponential, etc...

Working with a toy model shed some light on this problem. The goal was to take a user-defined function at two times along the x-axis, use the DTP to calculated the coefficients so that the solution could be calculated along the time axis. Once solved along the time axis, the test would be to extract the solution along two spatial locations, x_1 and x_2, and re-calculate the spectra using the DPP. If the solution is robust, the original user-defined functions should be reproducable using the DPP spectra to reconstruct along the spatial domain. Early attempts to perform this robust test of these methods failed.

Careful inspection of the reconstructed results revealed that the methods begin to falter as the solution is projected towards the edges of the domains, both time and space. The reason stems from the fact that the correct solution requires a summation over all modes, from $n = 1$ up to $n = \infty$. When fewer modes are used, taking N_{max} from the Nyquist limit, the solution begins to fail in its reconstruction as it approaches the far edges of the domain. Further preventing the solution from matching predictions beyond the windows is the fact that harmonic functions repeat outside of thier stated domains, so a fucntion that would normally increase linearly in time, becomes a sawtooth function, suddenly dropping as t approaches its limits. This abrupt change in the solution prevents the toy model from easily tranferring spectra from the time domain to the spatial domain and then accurately transforming back to the time domain, for validation. To address this issue about crossing the domain windows, a linear term is added to both the time and space functions. The end result is 16 terms, requiring 8 spatial locations to fully solve for the spectral coefficients. In order to avoid a singular matrix inversion, by choosing 10 spatial locations instead of 8, when a particular matrix is close to being singular, the spectra obtained tend to vary significantly. Rather than throw out these results, for the set of 10 locations, every set of 8 locations is used to calculate the spectra, giving

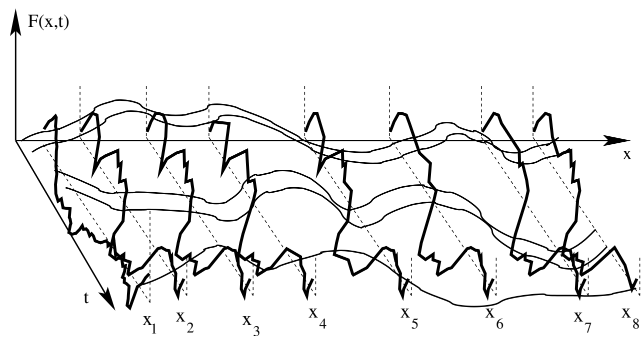


Fig. 9 Realistically, more than two positions will be used to fully calculate the coefficients, leading to better prediction in both time and space.

a set of results for the coefficients of size $\frac{1}{2}N!/8! = 45$, where N is the total number of locations taken. Taking the mean of the spectral results prevents any one result from skewing the data set.

$$
\begin{aligned}
F(x,t) = &\sum_{n=0}^{\infty} \left[A_n g(t)_{D,n} + B_n g(t)_{N,n} + E_n t + F_n\right] \\
&\times \left[C_n f(x)_{D,n} + D_n f(x)_{N,n} + G_n x + H_n\right] \\
F(x_1,t) = &\sum_{n=0}^{\infty} \left[AC_n g(t)_{D,n} f(x_1)_{D,n} + BC_n g(t)_{N,n} f(x_1)_{D,n}\right] + \\
&+ \left[AD_n g(t)_{D,n} f(x_1)_{N,n} + BD_n g(t)_{N,n} f(x_1)_{N,n}\right] + \\
&+ \left[EC_n\, t\, f(x_1)_{D,n} + ED_n\, t\, f(x_1)_{N,n}\right] + \\
&+ \left[AG_n\, x_1\, g(t)_{D,n} + BG_n\, x_1\, g(t)_{N,n}\right] + \\
&+ \left[FC_n\, f(x_1)_{N,n} + FD_n\, f(x_1)_{N,n}\right] + \\
&+ \left[AH_n\, g(t)_{D,n} + BH_n\, g(t)_{N,n}\right] + \\
&+ \left[EG_n\, x_1 t + EH_n\, t + FG_n\, x_1 + FH_n\right] \\
\alpha_m = &\oint F(x_1,t)_{data} g(t)_{D,m}\, d\, t
\end{aligned}
$$

$$
\begin{pmatrix} \alpha_n & \gamma_n & \cdots & \\ \beta_n & \Delta_n & & \\ \vdots & & \ddots & \\ & & & (4x4) \end{pmatrix} = \begin{pmatrix} AC_n & AD_n & \\ BC_n & BD_n & \\ & & \ddots \end{pmatrix} \begin{pmatrix} f(x_1)_{D,n} & f(x_2)_{D,n} & \\ f(x_1)_{N,n} & f(x_2)_{N,n} & \\ & & \ddots \end{pmatrix}
$$

$$
\begin{pmatrix} AC_n & AD_n & \cdots & \\ BC_n & BD_n & & \\ \vdots & & \ddots & \\ & & & (4x4) \end{pmatrix} = \begin{pmatrix} \alpha_n & \gamma_n & \\ \beta_n & \Delta_n & \\ & & \ddots \end{pmatrix} \begin{pmatrix} f(x_1)_{D,n} & f(x_2)_{D,n} & \\ f(x_1)_{N,n} & f(x_2)_{N,n} & \\ & & \ddots \end{pmatrix}^{-1}
$$

The Chesapeake Bay remains, and will so for a while, a work in progress. This latest revelation, that one can use a small number of locations to monitor the entire Bay by taking sufficient time windows presents a hopeful outlook for future signal characterization of the Bay. The utility to the COMSOL community is more towards application. By using a high-performance computing application like COMSOL, scientists are capable of producing results which are pushing our traditional approaches to older problems, such as signal processing. Problems encountered with the Chesapeake have simply been unprecedented in the past due to a lack of confidence in the solutions computed for the spatial domain. Through COMSOL, one must re-think the entire avenue set down by the one dimensional (time-based) signal processing community.

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